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Keith A Benedict†‡§

† Department of Theoretical Physics, The Schuster Laboratory, The University of Manchester M13 9PL, UK
‡ Department of Physics, The Blackett Laboratory, Imperial College, London SW7 2BZ, UK

Abstract. The zero-temperature helicity modulus tensor of the frustrated XY-model is calculated by analysing the spectrum of the Hessian matrix for the flux-density wave state near the centre of the Brillouin zone for the lowest band. The eigenproblem for the Hessian is formulated in terms of transfer matrices. Expressions for the derivatives of the lowest band with respect to the wavevector are found in terms of matrix traces which are then evaluated.

1. Introduction

The frustrated XY-model [1] is used to describe the statistical mechanics of Josephson junction arrays and granular superconductors. In the absence of disorder the model consists of a regular square lattice of superconducting grains embedded in a normal matrix. Each grain has an associated angular variable, θ_i, which is the phase of the superconducting order parameter or gap function. The intergrain coupling between phases is described by the effective free energy

$$H = -J(T) \sum_{\langle i,j \rangle} \cos(\phi_{i,j}) \tag{1.1}$$

where $\phi_{i,j} = \theta_i - \theta_j - A_{i,j}$, the twist factors are given by

$$A_{i,j} = \frac{2\pi}{\Phi_0} \int_i^j \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l}.$$

$\boldsymbol{A}(\boldsymbol{r})$ is the vector potential of an applied magnetic field and Φ_0 is the appropriate flux quantum. The sum of the twists around a plaquette, α, is given by

$$\sum_{\{i,j\} \in \partial\alpha} A_{i,j} = 2\pi f_\alpha$$

where f_α is the flux threading the plaquette in units of the flux quantum. In the following we will assume that the flux is the same on each plaquette of the lattice.

§ Present address: Department of Physics, The University, Nottingham NG7 2RD, UK.

Halsey [3] has proposed a state which is a minimum of the effective Hamiltonian
(1.1) as the ground state of the model for $\frac{1}{3} < f < \frac{1}{2}$. The form of this state is a
flux-density wave. In a companion paper to this [2] the fluctuation modes around this
ground state are discussed. Here I will calculate the helicity modulus tensor which
describes the susceptibility of the state to very long wavelength twist deformations of
the phase ordering. This quantity is related to the experimentally accessible complex
impedance $Z(\omega)$ of the network in coupled inductor experiments [4]. The kinetic
inductance is given by

$$L_{\mu,\nu} \sim \lim_{\omega \to 0} \frac{\mathrm{Im}[Z_{\mu,\nu}(\omega)]}{\omega}. \tag{1.2}$$

This is related to the helicity modulus by

$$L_{\mu,\nu} = \left(\frac{\hbar}{2e}\right)^2 (\Gamma^{-1})_{\mu,\nu} \tag{1.3}$$

where Γ^{-1} is the matrix inverse of the helicity modulus tensor.

Fluctuations around the ground state are described by the normal modes of the
Hessian or stability matrix

$$\mathbf{M}_{x,y;x',y'} = \left(\frac{\partial^2 H}{\partial \theta_{x,y} \partial \theta_{x',y'}}\right)_{\bar{\theta}}. \tag{1.4}$$

If $f = p/q$ then this matrix is periodic and Bloch's theorem can be used to reduce it
to a $q \times q$ matrix, the eigenvectors of the full matrix being

$$\psi_{x,y} = e^{ik_\parallel(x-y)} e^{ik_\perp(x+y)} C^{(\nu)}_{x+y}$$

where C is the eigenvector of the reduced Hessian satisfying $C_{z+q} = C_z$ and $(k_\parallel, k_\perp)$
takes values in the anisotropic Brillouin zone $-\pi/2 \le k_\parallel \le \pi/2$, $-\pi/2q \le k_\perp \le \pi/2q$.

The reduced eigenvalue equation has the form

$$(A_i + A_{i+1})C_i - xz^* A_i C_{i-1} - xz A_{i+1} C_{i+1} = \tfrac{1}{2}\Lambda C_i \tag{1.5}$$

where $x = \cos(k_\parallel)$, $z = e^{ik_\perp}$ and the A_is are a set of q characteristic energies for the
ground state (see [2] for details). The values of the A_is only depend on q, different
values of p permute the subscripts i.

2. Transfer matrix formulation of the eigenvalue problem

The eigenvalue equation (1.5) can be rewritten in matrix form as follows

$$\begin{bmatrix} C_{i+1} \\ C_i \end{bmatrix} = \frac{1}{xz A_{i+1}} \begin{bmatrix} A_i + A_{i+1} - \frac{1}{2}\Lambda & -xz^* A_i \\ xz A_{i+1} & 0 \end{bmatrix} \begin{bmatrix} C_i \\ C_{i-1} \end{bmatrix}. \tag{2.1}$$

Let

$$\mathbf{T}_i(\Lambda; x, z) = \begin{bmatrix} A_i + A_{i+1} - \frac{1}{2}\Lambda & -xz^* A_i \\ xz A_{i+1} & 0 \end{bmatrix} \tag{2.2}$$

and define

$$\mathbf{Q}(\Lambda; x, z) = \prod_{i=1}^{q} \mathbf{T}_i(\Lambda; x, z) \tag{2.3}$$

where the product sign implies ordering the lowest indexed matrix to the right. Clearly

$$\begin{bmatrix} C_{q+1} \\ C_q \end{bmatrix} = \frac{1}{\xi(xz)^q} \mathbf{Q}(\Lambda; x, z) \begin{bmatrix} C_1 \\ C_0 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_0 \end{bmatrix} \tag{2.4}$$

where $\xi = \prod_{i=1}^{q} A_i$. The matrix $\mathbf{U} = \mathbf{Q}/\xi(xz)^q$ must have a unit eigenvalue and therefore satisfy

$$1 - \mathrm{Tr}\,\mathbf{U} + \mathrm{Det}\,\mathbf{U} = 0. \tag{2.5}$$

Now $\mathrm{Tr}\,\mathbf{U} = \mathrm{Tr}\,\mathbf{Q}/\xi(xz)^q$, $\mathrm{Det}\,\mathbf{U} = \mathrm{Det}\,\mathbf{Q}/\xi^2(xz)^{2q}$ and

$$\begin{aligned} \mathrm{Det}\,\mathbf{Q} &= \prod_{i=1}^{q} \mathrm{Det}\,\mathbf{T}_i \\ &= x^{2q} \prod_{i=1}^{q} A_i A_{i+1} \\ &= x^{2q}\xi^2 \end{aligned} \tag{2.6}$$

so the eigenvalue condition (2.5) can be rewritten as

$$S(\Lambda; x, y) = \mathrm{Tr}\,\mathbf{Q}(\Lambda; x, z) = 2\xi x^q y \tag{2.7}$$

where $y = \frac{1}{2}(z^q + (z^*)^q) = \cos(qk_\perp)$. Equation (2.7) will be used as the fundamental condition that Λ be an eigenvalue of the reduced Hessian. In the next section it will be shown that $S(\Lambda; x, y)$ is independent of the variable y.

3. Proof that $S(\Lambda)$ is independent of y

The transfer matrix $\mathbf{T}$ can be decomposed as follows

$$\mathbf{T}_i(\Lambda; x, z) = \lambda_i(\Lambda)\mathbf{g} + x\mathbf{h}_i(z) \tag{3.1}$$

where

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{h}_i(z) = \begin{pmatrix} 0 & -z^* A_i \\ z A_{i+1} & 0 \end{pmatrix} \tag{3.2}$$

and $\lambda_i(\Lambda) = A_i + A_{i+1} - \frac{1}{2}\Lambda$.

It will be convenient to introduce the following notation. Consider a word consisting of an ordered string of q characters chosen from $\{g, h\}$. Each word represents the trace of the product of the corresponding terms in the decomposition (3.1). For example for $q = 2$ there would be the four words

$$[gg] = \lambda_1 \lambda_2 \mathrm{Tr}(\mathbf{g}^2) \qquad [gh] = x\lambda_2 \mathrm{Tr}(\mathbf{g}\mathbf{h}_1)$$

$$[hg] = x\lambda_1 \mathrm{Tr}(\mathbf{h}_2\mathbf{g}) \qquad [hh] = x^2 \mathrm{Tr}(\mathbf{h}_2\mathbf{h}_1). \tag{3.3}$$

Clearly $S(\Lambda; x, y)$ is given by the sum of all possible words.

Since $\mathbf{h}_i$ is purely off-diagonal and $\mathbf{g}$ is diagonal it follows that any product of an odd number of $\mathbf{h}$s will be off-diagonal and therefore traceless, consequently any word with an odd number of hs will not contribute to S. This immediately implies that $S(\Lambda; x, y)$ is an even function of x.

Consider now a word in which the substring ghg appears. This corresponds to

$$x\lambda_{i-1}\lambda_{i+1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -z^*A_i \\ zA_{i+1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \tag{3.4}$$

hence any word with isolated hs does not contribute either.

Now we consider partitioning a given word, which does not vanish for either of the above reasons, such that each partition contains an even number of hs, no partition can itself be decomposed into allowed partitions and no division falls between two gs. For example $[gg|hh|ggg|hh|hgh|hh|hggggh|hh|hh]$ is correctly partitioned. This partitioning is unique. If we consider the matrix product consisting of the subword within one partition there are three possible cases. If the subword contains no hs then the matrix product is obviously independent of z. If the subword is hh then the product is

$$x^2 \begin{bmatrix} 0 & -z^*A_i \\ zA_{i+1} & 0 \end{bmatrix} \begin{bmatrix} 0 & -z^*A_{i-1} \\ zA_i & 0 \end{bmatrix} = x^2 \begin{bmatrix} -A_i^2 & 0 \\ 0 & -A_{i-1}A_{i+1} \end{bmatrix} \tag{3.5}$$

which is independent of z. Finally if the subword is a string of gs with an h at either end then, using the idempotence of $\mathbf{g}$ and ignoring the λ factors, we get

$$\begin{bmatrix} 0 & -z^*A_i \\ zA_{i+1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -z^*A_j \\ zA_{j+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -A_{i+1}A_j \end{bmatrix} \tag{3.6}$$

which is independent of z.

Hence we conclude that $S(\Lambda, x, y)$ depends only on Λ and on x^2, not on z, y or any other function of $k_\perp$. Since $\mathbf{Q}$ is independent of z we can choose to construct it from $\mathbf{T}$ matrices evaluated at any convenient value of z, subject to the constraint $|z| = 1$; the most sensible choice is $z = 1$ so that the eigenvalue condition can now be written as

$$S(\Lambda, x) = 2\xi x^q y \tag{3.7}$$

where $S(\Lambda, x) = \mathrm{Tr} \prod_{i=1}^q \mathbf{T}_i(\Lambda, x, 1)$.

4. Local analysis near the zone centre

Close to the centre of the Brillouin zone we can approximate the lowest band by

$$\omega^{(0)}(\mathbf{k}) = \omega^{(0)}(0) + \left(k_\mu \frac{\partial \omega^{(0)}(\mathbf{k})}{\partial k_\mu}\right)_{\mathbf{k}=0} + \tfrac{1}{2} k_\mu \Gamma_{\mu,\nu} k_\nu + \mathrm{O}(k^3) \tag{4.1}$$

where

$$\Gamma_{\mu,\nu} = \left(\frac{\partial^2 \omega^{(0)}(\mathbf{k})}{\partial k_\mu \partial k_\nu}\right)_{\mathbf{k}=0}. \tag{4.2}$$

From the invariance of the Hamiltonian under global phase rotation and the fact that the state is a minimum of the Hamiltonian (and that the Hessian is therefore non-negative) we know that $\omega^{(0)}(0) = 0$. The first-order terms vanish at the zone centre by reason of symmetry (only even functions of k appear in the eigenvalue equations). The second-order terms can be written in the form

$$\left(\frac{\partial^2 \omega^{(0)}(k)}{\partial k_\parallel^2}\right)_{k=0} = -J\left(\frac{\partial \Lambda^{(0)}(x,1)}{\partial x}\right)_{x=1}$$

$$\left(\frac{\partial^2 \omega^{(0)}(k)}{\partial k_\perp^2}\right)_{k=0} = -Jq^2\left(\frac{\partial \Lambda^{(0)}(1,y)}{\partial y}\right)_{y=1} \tag{4.3}$$

$$\left(\frac{\partial^2 \omega^{(0)}(k)}{\partial k_\parallel \partial k_\perp}\right)_{k=0} = 0.$$

By taking derivatives of equation (3.7) it can be shown (appendix 1) that

$$\Gamma_\parallel = J\left(\frac{S_x(0,1) - 2\xi q}{S_\Lambda(0,1)}\right) \qquad \Gamma_\perp = J\left(\frac{-2\xi q^2}{S_\Lambda(0,1)}\right) \tag{4.4}$$

where

$$S_x(\Lambda, x) = \frac{\partial S(\Lambda, x)}{\partial x} \qquad S_\Lambda(\Lambda, x) = \frac{\partial S(\Lambda, x)}{\partial \Lambda}. \tag{4.5}$$

The modes corresponding to the lowest band near the zone centre are long-wavelength uniform rotations of the phase angles θ_i hence the coefficients $\Gamma_{\mu,\nu}$ characterize the energy cost of externally applying such phase rotations to the system. These are usually referred to as the helicity moduli of the system (at finite temperature it is the free energy cost which enters the definition) [4].

5. Calculation of the derivatives of S

In this section we will evaluate the two derivatives $S_\Lambda(0,1)$ and $S_x(0,1)$.

5.1. Calculation of $S_\Lambda(0,1)$

Differentiation of (3.7) with respect to Λ gives

$$S_\Lambda(0,1) = -\frac{1}{2}\mathrm{Tr}\left\{\mathbf{t}_{q,2}\mathbf{g} + \sum_{i=2}^{q-1}\mathbf{t}_{q,i+1}\mathbf{g}\mathbf{t}_{i-1,1} + \mathbf{g}\mathbf{t}_{q-1,1}\right\} \tag{5.1}$$

where

$$\mathbf{t}_{k,j} = \prod_{i=j}^{k}\mathbf{T}_i(0;1,1) \tag{5.2}$$

and

$$\left(\frac{\partial \mathbf{T}_i(\Lambda,1)}{\partial \Lambda}\right)_{\Lambda=0} = -\tfrac{1}{2}\mathbf{g} = -\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}. \tag{5.3}$$

It can be shown (appendix 2) that

$$\mathbf{t}_{j,k} = \left(\prod_{i=j}^{k} A_i\right) \left\{ \frac{1}{A_i} \mathbf{r}_{k+1,j} + A_{k+1} l_{k,j+1} \mathbf{w} \right\} \tag{5.4}$$

where $l_{k,j}$, $\mathbf{r}_{k,j}$ and $\mathbf{w}$ are defined in (A2.2). Evaluation of (5.1.1) using the trace formulæ (A3.1) yields

$$S_{\mathrm{A}}(0,1) = -\tfrac{1}{2}\xi q L_-(q) \tag{5.5}$$

where

$$L_-(q) = \sum_{i=1}^{q} \frac{1}{A_i} \tag{5.6}$$

which is a function of q only since it is a sum over all subscripts.

5.2. Calculation of $S_x(0,1)$

Differentiation of (3.7) with respect to x gives

$$S_x(0,1) = \mathrm{Tr}\,\mathbf{t}_{q,2}\mathbf{h}_1 + \mathrm{Tr}\sum_{i=2}^{q-1}\mathbf{t}_{q,i+1}\mathbf{h}_i\mathbf{t}_{i-1,1} + \mathrm{Tr}\mathbf{h}_q\mathbf{t}_{q-1,1} \tag{5.7}$$

where

$$\mathbf{h}_i = \left(\frac{\partial \mathbf{T}_i(0;x,1)}{\partial x}\right)_{x=1} = \begin{bmatrix} 0 & -A_i \\ A_{i+1} & 0 \end{bmatrix}. \tag{5.8}$$

Again employing the result of appendix 2 gives, with the use of (A3.2),

$$S_x(0,1) = 2\xi\left(q - L_+(q)L_-(q)\right) \tag{5.9}$$

where

$$L_+(q) = \sum_{i=1}^{q} A_i. \tag{5.10}$$

6. The helicity moduli

Using the results of the previous section gives the following for the two non-zero components of the helicity modulus tensor

$$\Gamma_{\|,\|} = \frac{4L_+(q)}{q} \qquad \Gamma_{\perp,\perp} = \frac{4q}{L_-(q)}. \tag{6.1}$$

$L_{\pm}$ have the form

$$L_+ = \begin{cases} 1 + 2\sum_{i=1}^{(q-1)/2}\cos(i/\pi/q) & q\ \text{odd} \\ 2\sum_{i=1}^{q/2}\cos(i/\pi/q + \pi/2q) & q\ \text{even} \end{cases} \tag{6.2a}$$

and

$$L_- = \begin{cases} 1 + 2\sum_{i=1}^{(q-1)/2} \sec(i\pi/q) & q \text{ odd} \\ 2\sum_{i=1}^{q/2} \sec(i\pi/q + \pi/2q) & q \text{ even.} \end{cases} \tag{6.2b}$$

For all q, L_+ has the form

$$L_+(q) = \left(\frac{2\sin(\pi/2q)}{1 - \cos(\pi/q)} \right) \tag{6.4}$$

hence

$$\Gamma_{\|,\|} = \frac{8J\sin(\pi/2q)}{q(1 - \cos(\pi/q))}. \tag{6.5}$$

In the asymptotic limit, $q \to \infty$, we can write

$$L_+ \sim \frac{2q}{\pi} \qquad L_- \sim \frac{2q}{\pi}\ln\left(\frac{4q}{\pi}\right) \tag{6.6}$$

yielding for the helicity moduli

$$\Gamma_{\|,\|} \sim \frac{8J}{\pi} \qquad \Gamma_{\perp,\perp} \sim \frac{2\pi J}{\ln(q)}. \tag{6.7}$$

7. Discussion

The non-vanishing components of the helicity modulus tensor have been calculated for the frustrated XY-model in the 'flux-density wave' states which are believed to be the ground states for some rational values of the flux parameter $f = p/q$ in the range $\frac{1}{3} < f < \frac{1}{2}$. Explicit expressions have been given in terms of sums of the bond energies which characterize the state. In addition asymptotic forms, valid for large q have been given. The relevance of the large q limit is discussed further in [2]. It should be noted that this quantity depends only on the denominator q. The fluctuations described by these parameters are analogous to the spin-wave excitations in magnetic systems and are therefore responsible for the absence of long-range order of the phases at any finite temperature [5] and hence the removal of the mean field phase transition. The actual transition in this system, as in the ordinary XY-model, is mediated by the topological excitations (vortices and domain walls) which are not accessible to the simple spin-wave theory. However, the result of this paper shows that fluctuations perpendicular to the staircase directions are stronger for higher q. This is in agreement with numerical simulations [1] which show that the critical temperature is depressed by increasing q although the mechanism for this depression probably has more to do with the decrease in energy cost for the formation of topological excitations.

Acknowledgments

Appendix 1. Derivative formulæ

The eigenvalue equation (3.7) is

$$S\big(\Lambda(x,y),x\big) = 2\xi x^q y \tag{A1.1}$$

differentiating with respect to x gives

$$\frac{\partial \Lambda(x,y)}{\partial x}\left(\frac{\partial S(\Lambda,x)}{\partial \Lambda}\right)_{\Lambda=\Lambda(x,y)} + \left(\frac{\partial S(\Lambda,x)}{\partial x}\right)_{\Lambda=\Lambda(x,y)} = 2q\xi x^{q-1}y \tag{A1.2}$$

hence

$$\frac{\partial \Lambda(x,y)}{\partial y} = \frac{2q\xi x^{q-1}y - S_x(\Lambda,x)}{S_\Lambda(\Lambda,x)}. \tag{A1.3}$$

Similarly differentiating (A1.1) with respect to y gives

$$\frac{\partial \Lambda(x,y)}{\partial y}\left(\frac{\partial S(\Lambda,x)}{\partial \Lambda}\right)_{\Lambda=\Lambda(x,y)} = 2\xi x^q \tag{A1.4}$$

which, on rearrangement, yields

$$\frac{\partial \Lambda(x,y)}{\partial y} = \frac{2\xi x^q}{S_\Lambda(\Lambda,x)}. \tag{A1.5}$$

Appendix 2. Proof of the expression for $\mathbf{t}_{k,j}$

We wish to prove that

$$\mathbf{t}_{k,j} = \left(\prod_{i=j}^{k} A_i\right)\left\{\frac{1}{A_i}\mathbf{r}_{k+1,j} + A_{k+1}l_{k,j+1}\mathbf{w}\right\} \tag{A2.1}$$

where

$$l_{k,j} = \sum_{i=j}^{k} \frac{1}{A_i} \tag{A2.2a}$$

$$\mathbf{r}_{k,j} = \begin{bmatrix} A_k + A_j & -A_j \\ A_k & 0 \end{bmatrix} \tag{A2.2b}$$

and

$$\mathbf{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{A2.2c}$$

Clearly $\mathbf{t}_{i,i} = \mathbf{T}_i(0;1,1) = \mathbf{r}_{i+1,i}$ and it is easy to verify that

$$\mathbf{t}_{i+1,i} = A_{i+1}A_i\left\{\frac{1}{A_i}\mathbf{r}_{i+2,i} + \frac{A_{i+2}}{A_{i+1}}\mathbf{w}\right\}. \tag{A2.3}$$

If we assume that expression (A2.1) is correct for $\mathbf{t}_{k,j}$ then using the relations $\mathbf{r}_{i,j}\mathbf{r}_{j,k} = A_j\mathbf{r}_{i,k} + A_i A_k\mathbf{w}$, $\mathbf{r}_{k,j}\mathbf{w} = A_k\mathbf{w}$ and $\mathbf{wr}_{k,j} = A_j\mathbf{w}$ gives

$$\mathbf{t}_{k+1,j} = \mathbf{T}_{k+1}(0;1,1)\mathbf{t}_{k,j}$$

$$= \mathbf{r}_{k+2,k+1}\left(\prod_{i=j}^{k}A_i\right)\left\{\frac{1}{A_j}\mathbf{r}_{k+1,j} + A_{k+1}l_{k,j+1}\mathbf{w}\right\}$$

$$= \left(\prod_{i=j}^{k}A_i\right)\left\{\frac{1}{A_j}\left[A_{k+1}\mathbf{r}_{k+2,j} + A_{k+2}A_j\mathbf{w}\right] + A_{k+1}l_{k,j+1}A_{k+2}\mathbf{w}\right\}$$

$$= \left(\prod_{i=j}^{k+1}A_i\right)\left\{\frac{1}{A_j}\mathbf{r}_{k+2,j} + A_{k+2}l_{k+1,j+1}\mathbf{w}\right\} \tag{A2.4}$$

which corresponds to (A2.1) for $\mathbf{t}_{k+1,j}$. Hence, by induction, since the expression is true for $\mathbf{t}_{i+1,i}$ then it is true for all $\mathbf{t}_{j,i}$.

As a check we can use the formula to evaluate $S(0,1)$

$$S(0,1) = \mathrm{Tr}(\mathbf{t}_{q,1})$$

$$= \left(\prod_{i=1}^{q}A_i\right)\mathrm{Tr}\left\{\frac{1}{A_q}\mathbf{r}_{1,1} + A_1 l_{q,2}\mathbf{w}\right\}$$

$$\equiv 2\xi \tag{A2.5}$$

which agrees with equation (3.7).

Appendix 3. Trace formulæ

The following are trace formulæ used in the derivation of results (5.1.5) and (5.2.3):

$$\mathrm{Tr}[\mathbf{r}_{1,i+1}\mathbf{g}\mathbf{r}_{i,1}] = A_i A_{i+1} + A_i A_1 + A_1 A_{i+1}$$

$$\mathrm{Tr}[\mathbf{wg}\mathbf{r}_{i,1}] = A_i$$

$$\mathrm{Tr}[\mathbf{r}_{1,i+1}\mathbf{gw}] = A_{i+1}$$

$$\mathrm{Tr}[\mathbf{wgw}] = 0$$

$$\mathrm{Tr}[\mathbf{g}\mathbf{r}_{q,1}] = A_1 + A_q$$

$$\mathrm{Tr}[\mathbf{r}_{1,2}\mathbf{g}] = A_1 + A_2$$

$$\mathrm{Tr}[\mathbf{gw}] = 1$$

$$\mathrm{Tr}[\mathbf{wg}] = 1 \tag{A3.1}$$

$$\mathrm{Tr}[\mathbf{r}_{1,i+1}\mathbf{h}_i\mathbf{r}_{i,1}] = -A_i^2(A_1 + A_{i+1}) - A_{i+1}^2(A_1 + A_i)$$

$$\mathrm{Tr}[\mathbf{w}\mathbf{h}_i\mathbf{r}_{i,1}] = -A_i(A_i + A_{i+1})$$

$$\mathrm{Tr}[\mathbf{r}_{1,i+1}\mathbf{h}_i\mathbf{w}] = -A_{i+1}(A_i + A_{i+1})$$

$$\mathrm{Tr}[\mathbf{w}\mathbf{h}_i\mathbf{w}] = 0$$

$$\mathrm{Tr}[\mathbf{h}_q\mathbf{r}_{q,1}] = -(A_1^2 + A_q^2)$$

$$\mathrm{Tr}[\mathbf{r}_{1,2}\mathbf{h}_1] = -(A_1^2 + A_2^2)$$

$$\mathrm{Tr}[\mathbf{h}_q\mathbf{w}] = -(A_1 + A_q)$$

$$\mathrm{Tr}[\mathbf{w}\mathbf{h}_1] = -(A_1 + A_2). \tag{A3.2}$$

References

[1]   Teitel S and Jayaprakash C 1983 *Phys. Rev. Lett.* **51** 1999–2002; 1983 *Phys. Rev.* B **27** 598–601
      Shih W Y, Ebner C and Stroud D 1984 *Phys. Rev.* B **30** 134–44
[2]   Benedict K A 1991 *J. Phys.: Condens. Matter* submitted
[3]   Halsey T C 1985 *Phys. Rev.* B **31** 5728–45
[4]   Lobb C J 1988 *Physica* B **15** 1–6
[5]   Mermin N D and Wagner H 1966 *Phys. Rev. Lett.* **17** 1133